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On Convergence of Shock-Capturing Schemes inside the Shocks Influence Area

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Godunov's taboo

There are no monotone finite-difference schemes (with smooth numerical flux functions) higher than the first order of approximation

Godunov S.K. A difference Method for Numerical Calculation of Discontinuous Solutions of the Equations of Hydrodynamics, *Mat. Sb.* 1959.

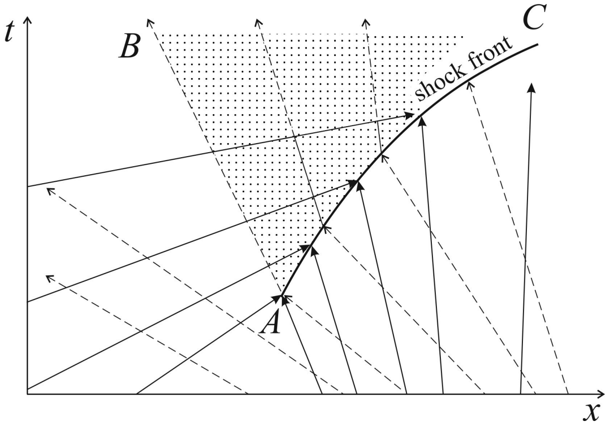
Attempts to overcome Godunov's taboo

NFC-like schemes (Nonlinear Flux Correction)

- *Goldin, Kalitkin, Shitova*, 1965.
- FCT, *Boris, Book*, 1975.
- *Kolgan*, 1978.
- MUSCL, *Van Leer*, 1979.
- TVD schemes, *Harten*, 1983.
- ENO schemes, *Harten, Osher*, 1987.
- CU schemes, *Tadmor*, 1990.
- WENO schemes, *Liu, Osher, Chan*, 1994; *Jiang, Shu*, 1996.
- CABARET schemes, *Samarskii, Goloviznin, Karabasov*, 1998, 2005.

Shock Influence Area for a Hyperbolic System of Two Conservation Laws

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0 \tag{1}$$



Fields of characteristics for a system of two conservation laws

Definition of Integral Convergence

Let's set the number $a \in \mathbb{R}$ and define the integrals

$$\boldsymbol{U}^a(t,x) = \int_x^a \boldsymbol{u}(t,y)dy, \quad \boldsymbol{V}_i^a(t,x) = \int_x^a \boldsymbol{v}_i(t,y)dy \tag{2}$$

Definition 1.

The sequence of difference solutions $\boldsymbol{v}_i(t,x)$ converges on the interval $[x,a] \subset \mathbb{R}$ with the Rth order ($0 < R \le 2$), to the exact solution $\boldsymbol{u}(t,x)$, if

$$\boldsymbol{V}_i^a(t,x) - \boldsymbol{U}^a(t,x) = \boldsymbol{C}\Delta_i^R + o\left(\Delta_i^R\right),$$

where the vector function $\boldsymbol{C}$ is independent of Δ_i.

$$\Delta_0 = \Delta, \quad \Delta_1 = \Delta/2, \quad \Delta_2 = \Delta/4$$

$$\delta \boldsymbol{V}_{i,i+1} = \boldsymbol{V}_i^a - \boldsymbol{V}_{i+1}^a = \boldsymbol{C}(\Delta_i^R - \Delta_{i+1}^R), \quad i = 0, 1.$$

$$\frac{|\delta \boldsymbol{V}_{1,2}|}{|\delta \boldsymbol{V}_{0,1}|} = \frac{\Delta_1^R - \Delta_2^R}{\Delta_0^R - \Delta_1^R} = \left(\frac{1}{2}\right)^R \quad \Rightarrow \quad R = \log_{1/2} \frac{|\delta \boldsymbol{V}_{1,2}|}{|\delta \boldsymbol{V}_{0,1}|} \tag{3}$$

Kovyrkina O.A., Ostapenko V.V. On the convergence of shock-capturing difference schemes, *Dokl. Math.* 2010.

Real Accuracy of NFC-schemes

1. **Ostapenko V.V.** On convergence of difference schemes behind of nonstationary shock, *Comp. Maths Math. Phys.* 1997.
2. **Casper J., Carpenter M.N.** Computational consideration for the simulation of shock-induced sound, *SIAM J. Sci. Comput.* 1998.
3. **Engquist B., Sjogreen B.** The convergence rate of finite difference schemes in the presence of shocks, *SIAM J. Numer. Anal.* 1998.
4. **Kovyrkina O.A., Ostapenko V.V.** On the convergence of shock-capturing difference schemes, *Dokl. Math.* 2010.
5. **Kovyrkina O.A., Ostapenko V.V.** On the practical accuracy of shock-capturing schemes, *Math. Models. Comput. Simul.* 2014.
6. **Kovyrkina O.A., Kudryavtsev A.N., Ostapenko V.V.** On real accuracy of WENO schemes at shock capturing calculations, *International conference «AMCA – 2014», Novosibirsk, Russia.*
7. **Mikhailov N.A.** The convergence order of WENO schemes behind a shock front, *Math. Models. Comput. Simul.* 2015.
8. **Kovyrkina O.A., Ostapenko V.V.** On monotonicity and accuracy of CABARET scheme for calculation of weak solutions with shocks, *Computational Technologies.* 2018. [In Russ.]
9. **Ladonkina M.E., Neklyudova O.A., Ostapenko V.V., Tishkin V.F.** On the accuracy of the discontinuous Galerkin method in calculation of shock waves, *Comp. Math. Math. Phys.* 2018.
10. **Bragin M.D., Rogov B.V.** On the accuracy of bicompact schemes as applied to computation of unsteady shock waves, *Comp. Math. Math. Phys.* 2020.

Why are Nonmonotonic Schemes with Higher Order Transfer the Rankine-Hugoniot Conditions through the Shock?

The **oscillations** arising on shock wave fronts in classical nonmonotonic schemes of high accuracy **keep information** about Fourier wave structure of the expansion of a discontinuous function in the vicinity of a strong discontinuity, which allows to these schemes with high accuracy transfer Rankine-Hugoniot conditions through the smeared shock wave fronts and conserve increased accuracy in the regions of shock wave influence.

At the same time NFC schemes, by smoothing these oscillations, lose this information, that leads to the decrease in their accuracy of transmitting the Rankine-Hugoniot conditions.

Alternative in the Theory of Finite-Difference Schemes

Is it not possible

to localize a shock wave front with higher accuracy
and, at the same time,
maintain an increased order of convergence
in the domain of influence
of the shock wave?

Combined Schemes

1. **Kovyrkina O.A., Ostapenko V.V.** On the construction of combined finite-difference schemes of high accuracy, *Dokl. Math.* 2018.
2. **Zyuzina N.A., Kovyrkina O.A., Ostapenko V.V.** Monotone finite-difference scheme preserving high accuracy in regions of shock influence, *Dokl. Math.* 2018.
3. **Ladonkina M.E., Nekliudova O.A., Ostapenko V.V., Tishkin V.F.** Combined DG scheme that maintains increased accuracy in areas of shock waves, *Dokl. Math.* 2019.
4. **Ladonkina M.E., Nekliudova O.A., Tishkin V.F.** Combined scheme based on Rusanov scheme and discontinuous Galerkin method, *AIP Conference Proceedings.* 2020
5. **Bragin M.D., Rogov B.V.** Combined monotone bicompact scheme with higher order accuracy in domains of influence of nonstationary shock waves, *Dokl. Math.* 2020.
6. **Bragin M.D., Rogov B.V.** Combined multidimensional bicompact scheme with higher order accuracy in domains of influence of nonstationary shock waves, *Dokl. Math.* 2020.
7. **Bragin M.D., Kovyrkina O.A., Ladonkina M.E., Ostapenko V.V., Tishkin V.F., Khandeeva N.A.** Combined Numerical Schemes, *Comp. Math. Math. Phys.* 2022.
8. **Chu S., Kovyrkina O.A., Kurganov A., Ostapenko V.V.** Experimental Convergence Rate Study (*prepared for print*)

Dam Break Problem for Shallow Water Equations

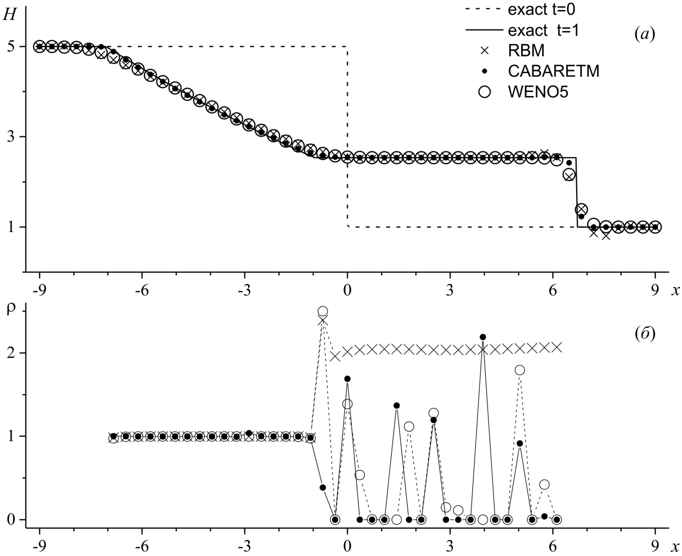
$$\begin{cases} h_t + q_x = 0 \\ q_t + (qu + gh^2/2)_x = 0 \end{cases} \tag{4}$$

$$H(x,0) = \begin{cases} 5, & x \le 0, \\ 1, & x > 0, \end{cases} \qquad q(x,0) = 0. \tag{5}$$

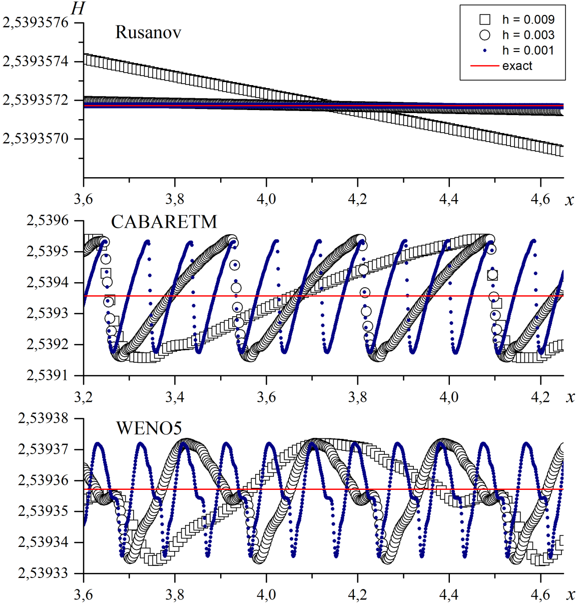
The solution of this problem consists of a shock wave propagating at the constant speed $D = 6.64$ and a centered depression wave with a constant flow region in between.

1. **Rusanov V.V.** Difference schemes of the third order of accuracy for the forward calculation of discontinuous solutions, *Dokl. Akad. Nauk SSSR*. 1968.
2. **Kovyrkina O. A., Ostapenko V. V.** Monotonicity of the CABARET scheme approximating a hyperbolic system of conservation laws, *Comput. Math. Math. Phys.* 2018.
3. **Jiang G.S., Shu C.W.** Efficient implementation of weighted ENO schemes, *J. Comput. Phys*. 1996.
4. **Kovyrkina O. A., Ostapenko V. V., Tishkin V. F.** On convergence of finite-difference shock-capturing schemes in the regions of shock waves influence, *Dokl. Math*. 2022.

Fluid Depth and Local Orders of Convergence
for the Rusanov, CABARETM and WENO5 Schemes at $T = 1$



Fluid Depth in the Constant-Flow Domain at $T = 1$



Conclusion

We show that in the Nonlinear Flux Correction (NFC) schemes CABARETM and WENO5 (in contrast to the Rusanov scheme), there is no local convergence of the difference solution inside the shock influence area when calculating the shallow water dam break problem. A similar result was obtained in [1] for the Discontinuous Galerkin method when calculating the classical Shu-Osher problem.

This is due to the fact that the numerical solutions obtained by these schemes have persistent oscillations inside the domain of a constant flow between a shock and a centered rarefaction wave. In this case, taking into account the Lax-Wendroff theorem, the numerical solutions obtained by the NFC schemes converge to the exact solution inside the shock influence area in the weak sense only.

1. **Ladonkina M.E., Neklyudova O.A., Tishkin V.F.** Solution of fluid dynamics problems by applying the Galerkin method with discontinuous basis functions, *Abstracts of papers of the International Conference on Modern Problems in Applied Mathematics and Computer Science*. Dubna, 2012.

THANK YOU FOR YOUR ATTENTION!

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